Lecture 8: Wave Eq. Pt 3 - Higher Dimensions

Model Problem: Sound Waves

- We could model the vibration of a drumhead on a bounded domain $\Omega \subseteq \mathbb{R}^2$ with $u(t,x)$ representing vertical displacement. We derive the wave equation as before

$$(A) \quad \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

Instead, let us jump to sound waves for $\mathbb{R}^3$.

- Let P denote the pressure in the air, with air density ρ and velocity field v. Since sound waves are relatively minute pressure fluctuations, we apply the adiabatic gas law

$$P = C\rho^{\gamma} \qquad \text{for } C, \gamma \text{ some physical constants}$$

Fix background atmospheric values P_0, ρ_0 and let us focus on the deviations

$$u = P - P_0 \qquad\qquad \sigma = \rho - \rho_0$$

applying the gas law,

$$1 + u/P_0 = \left(1 + \frac{\sigma}{\rho_0}\right)^{\gamma}$$

- We assume σ/ρ_0 to be very small, so that we may take a first-order Taylor approximation to the RHS

$$1 + \frac{u}{P_0} = 1 + \frac{\sigma}{\rho_0} \cdot \gamma \quad \text{or} \quad u = \frac{\gamma P_0}{\rho_0} \sigma \quad (1)$$

Recall lecture 5

- Conservation of mass relates ρ to v: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$

Since σ & v are assumed to be small, we can approximate $\rho \simeq \rho_0$, $\frac{\partial \sigma}{\partial t} \simeq \frac{\partial \rho}{\partial t}$ to get

$$\frac{\partial \sigma}{\partial t} + \rho_0 \nabla \cdot v = 0 \quad (2)$$

- to relate P to ρ & v, we use Euler's Force Equation

$$-\nabla P = \rho\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v \quad \text{or, for } P = P_0 + u \text{ \& } \rho = \rho_0 + \sigma:$$

$$-\nabla u = \rho_0 \frac{\partial v}{\partial t} + \text{Higher-order terms}$$

- We linearize: $-\nabla u = \rho_0 \frac{\partial v}{\partial t}$ (3)
- Next, we eliminate v: Substitute (1) into (2)

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{\gamma P_0} u\right) + \rho_0 \nabla \cdot v = 0$$

Differentiate & Simplify

$$\frac{\partial^2 u}{\partial t^2} = -\gamma P_0 \nabla \cdot \frac{\partial v}{\partial t}$$

by (3) $= -\gamma P_0 \nabla \cdot \left(-\frac{\nabla u}{\rho_0}\right) = \frac{\gamma P_0}{\rho_0} \Delta u$

giving

$$\frac{\partial^2 u}{\partial t^2} - \frac{\gamma P_0}{\rho_0} \Delta u = 0$$

the acoustic wave equation.

Integral Solution Formulas

Let us consider $\mathbb{R}^3$ and $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ (B)

We reduce to the 1D case

- For $f \in C^0(\mathbb{R}^3)$, define $\bar{f}(x;\rho) = \frac{1}{4\pi\rho} \int_{\partial B(x;\rho)} f(\omega)\, dS(\omega)$ for $x \in \mathbb{R}^3$, $\rho > 0$. Since f is continuous,

$$\lim_{\rho \to 0} \frac{\bar{f}(x;\rho)}{\rho} = f(x)$$

(Note that $\mathrm{Vol}(\partial B(x;\rho)) = 4\pi\rho^2$)

$\boxed{\text{Lemma 4.9: Darboux's Formula}}$ For $f \in C^2(\mathbb{R}^3)$

$$\frac{\partial^2}{\partial \rho^2} \bar{f}(x;\rho) = \Delta_x \bar{f}(x;\rho)$$

$\boxed{\text{Pf}}$ First, we standardize $\omega = x + \rho \cdot y$ for $y \in S^2$ and

$$\frac{1}{\rho}\bar{f}(x;\rho) = \frac{1}{4\pi\rho^2}\int_{\partial B(x;\rho)} f(\omega)\, dS = \frac{1}{4\pi}\int_{S^2} f(x+\rho y)\, dS(y)$$

and $\frac{\partial}{\partial \rho}\left[\frac{1}{4\pi}\int_{S^2} f(x+\rho \cdot y)\, dS(y)\right] = \frac{1}{4\pi}\int_{S^2} \nabla f(x+\rho y) \cdot y \, dS(y)$ → Before flipping page, what does this look like?

• Since y is the normal to S^2 at y,

$$\frac{1}{4\pi}\int_{S^2} \nabla f(x+\rho y)\cdot y\, dS(y) = \frac{1}{4\pi}\int_{B(0,1)} \Delta f(x+\rho y)\, dS(y)$$

$$= \frac{1}{4\pi\rho^2}\int_{B(x;\rho)} \Delta f(\omega)\, dS(\omega)$$

or

$$\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\bar{f}\right] = \frac{1}{4\pi\rho^2}\int_{B(x;\rho)} \Delta f(\omega)\, dS(\omega)$$

or

$$\frac{\partial}{\partial\rho}(\bar{f}) = \rho\left[\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\bar{f}\right) + \frac{1}{\rho^2}\bar{f}\right]$$

$$= \frac{1}{4\pi\rho^2}\int_{\partial B(x;\rho)} f(\omega)\, dS(\omega) + \frac{1}{4\pi\rho}\int_{B(x;\rho)} \Delta f(\tilde{\omega})\, d\tilde{\omega}$$

•) Repeating this sort of process gives

$$\frac{\partial^2}{\partial\rho^2}\bar{f}(x;\rho) = \cancel{\frac{1}{4\pi\rho}\int_{\partial B(x;\rho)}}\ \frac{1}{4\pi\rho}\frac{\partial}{\partial\rho}\int_{B(x;\rho)} \Delta f(\tilde{\omega})\, d\tilde{\omega}$$

$$= \frac{1}{4\pi\rho}\int_{\partial B(x;\rho)} \Delta f(\hat{\omega})\, dS(\omega)$$

•) Alternately, $\Delta_x \bar{f}(x;\rho) = \Delta_x\left[\frac{1}{4\pi\rho}\int_{S^2} f(x+\rho y)\, dS(y)\right]$

$$= \frac{1}{4\pi\rho}\int_{S^2} \Delta f(x+\rho y)\, dS(y)$$

$$= \frac{1}{4\pi\rho}\int_{\partial B(x;\rho)} \Delta f(\omega)\, dS(\omega)$$

> Chain rule $1/\rho^2$
> $\partial B(x;\rho)$ from S^2 gives ρ^2

□

•) This lemma relates a 3D-object (Δ_x) to a one-dimensional equation

Thm 4.10 Kirchhoff's Integral Formula

For $u \in C^2([0,\infty)\times\mathbb{R}^3)$, suppose that $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ under initial conditions $u|_{t=0} = g$, $\partial_t u|_{t=0} = h$.

Then, $u(t,x) = \partial_t \bar{g}(x;t) + \bar{h}(x;t)$

$\boxed{\text{Pf}}$ Define $\bar{u}(t,x;\rho) = \frac{1}{4\pi\rho} \int_{\partial B(x;\rho)} u(t,\omega)\, dS(\omega)$

By the Leibniz Rule & the wave equation

$$\frac{\partial^2}{\partial t^2} \bar{u}(t,x;\rho) = \frac{1}{4\pi\rho} \int_{\partial B(x;\rho)} \Delta u(t,\omega)\, dS(\omega)$$

$= \Delta_x \bar{u}(t,x;\rho)$ as we computed in the lemma above.

The lemma itself provides $\Delta_x \bar{u} = \partial^2/\partial\rho^2\, \bar{u}$, so

$$\left(\partial^2/\partial t^2 - \partial^2/\partial\rho^2 \right) \bar{u}(t,x;\rho) = 0$$

This is the 1-D wave equation. Let us then derive initial conditions

$$\bar{u}(0,x;\rho) = \frac{1}{4\pi\rho} \int_{\partial B(x;\rho)} u(0,\omega)\, dS(\omega) = \frac{1}{4\pi\rho} \int_{\partial B(x;\rho)} g(\omega)\, dS(\omega)$$

$$= \bar{g}(x;\rho)$$

Similarly, $\partial_t \bar{u}(0,x;\rho) = \bar{h}(x;\rho)$

and by definition, $\bar{u}(t,x;0) = 0$.

As in the 1-D case, we conclude that the unique solution is given by an odd extension of $\bar{g}, \bar{h}$ to $\rho \in \mathbb{R}$ and

$$\bar{u}(t,x;\rho) = \frac{1}{2}\left[\bar{g}(x;\rho+t) + \bar{g}(x;\rho-t)\right] + \frac{1}{2}\int_{\rho-t}^{\rho+t} \bar{h}(x;\tau)\, d\tau$$

Since $u(t,x) = \lim_{\rho\to 0} \frac{1}{\rho} \bar{u}(t,x;\rho)$, we can recover u.

For evaluation, write

$$\bar{u}(t,x;\rho) = \frac{1}{2}\left[\bar{g}(x;t+\rho) - \bar{g}(x;t-\rho)\right] + \frac{1}{2}\int_{t-\rho}^{t+\rho} \bar{h}(x;\tau)\, d\tau$$

and

$$\lim_{\rho\to 0} \frac{\bar{g}(x;t+\rho) - \bar{g}(x;t-\rho)}{2\rho} + \frac{1}{2\rho}\int_{t-\rho}^{t+\rho} \bar{h}(x;\tau)\, d\tau$$

$$= \partial_t \bar{g}(x;t) + \bar{h}(x;t) \qquad \square$$

•) Kirchoff's Formula exhibits the 3D, Strict form of Huygen's Principle. It shows that the range of influence of a point (t_0, x_0) is the forward light cone

$$T_+(t_0, x_0) = \{(t,x);\ t>t_0,\ |x-x_0| = t-t_0\}$$

⇝ Why is it Strict: this equality ↑ In our 1D case, we said that ~~that~~ the influence occured within some range, this gives it in a sharp space.

⇝ this more accurately describes why soundwaves are "Sharp", and why we hear a sharp wave front from clapping/noise. In 1D, it would be more sustained.

• The Spherical Averaging trick works in higher odd dimensions, but not even ones. For even dimensions, we derive them from known odd-dimension solutions by the ~~Metod~~ Method of Descent. We shall do this for $\mathbb{R}^2$.

• suppose $u \in C^2([0,\infty)\times\mathbb{R}^2)$ solves the wave equation with Initial Conditions

$$u|_{t=0} = g\ ;\quad \partial_t u|_{t=0} = h \qquad \text{for } g, h: \mathbb{R}^2 \to \mathbb{R}$$

$\boxed{\text{Th}^{m}\ 4.11}$ Poisson's Integral Formula ⇝ $B_2(0,1) \subseteq \mathbb{R}^2$

For $u \in C^2$, $g \in C^2$, $h \in C^1$ as above,

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_{B_2(0,1)} \frac{g(x-ty)}{\sqrt{1-|y|^2}}\,dy\right) + \frac{t}{2\pi}\int_{B_2(0,1)} \frac{h(x-ty)}{\sqrt{1-|y|^2}}\,dy$$

$\boxed{\text{Pf}}$ First, extend g & h to be functions of $\mathbb{R}^3$ independent of x_3. Because we act in $\mathbb{R}^3$,

$$\bar{g}(x;\rho) = \frac{\rho}{4\pi}\int_{S^2} g(x_1+\rho y_1, x_2+\rho y_2)\,dS(y)$$

As g, h are independent of x_3, we may reduce to the upper hemisphere by symmetry. We use polar coordinates

$$y = (r\cos(\theta),\ r\sin(\theta),\ \sqrt{1-r^2})$$

over which $dS = \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$

The Spherical average is then

$$\bar{g}(x;\rho) = \rho/2\pi \int_0^{2\pi}\int_0^1 \frac{g(x_1+\rho r\cos(\theta),\, x_2+\rho r\sin(\theta))}{\sqrt{1-r^2}}\, r\, dr\, d\theta$$

$$= \rho/2\pi \int_{B_2(0;1)} \frac{g(x+\rho y)}{\sqrt{1-|y|^2}}\, dy$$

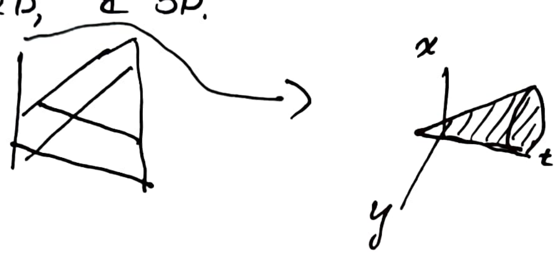
Similarly, derive $\bar{h}$ and place $\bar{g}$ & $\bar{h}$ into Kirchoff's formula. $\square$

•) In two dimensions, the range of influence is the solid region bounded by the light cone.

⇝ Compare 1D, 2D, & 3D.

↳

3D ⇝ light cone is 4D.



Energy & Uniqueness

- We present a uniqueness argument suitable for all dimensions based on energy
- First, look back on the string fixed on both ends and let $u \in C^2([0,\infty)\times[0,\ell])$ be the displacement.

Recall our linear density ρ, and our discretization to the segment Δx at x_j

Kinetic Energy $= \frac{1}{2}(\text{mass})(\text{velocity})^2$

KE $= \frac{1}{2}(\rho\Delta x)\left(\frac{\partial u}{\partial t}(x_j)\right)^2$ on this segment

or $\tilde{\mathcal{E}}_K = \sum_{j=0}^{n-1} \frac{1}{2}(\rho\Delta x)\left(\frac{\partial u}{\partial t}\right)^2$ and pushing $n \to \infty$

$$\mathcal{E}_K = \rho/2 \int_0^{\ell} \left(\frac{\partial u}{\partial t}\right)^2 dx$$

The potential energy can be calculated as the energy required to move the string from zero displacement to $u(t,\cdot)$.

We represent this process by scaling to $su(t,\cdot)$ for $s \in [0,1]$.

Since $\Delta F(t,x_j) = T\sin(\alpha_j) + T\sin(\beta_j) = \frac{T}{\Delta x}[u(t,x_{j+1}) + u(t,x_{j-1}) - 2u(t,x_j)]$, the force also scales with s. The work to shift $s \to s+\Delta s$ at x_j is then ~~$s\Delta F(x_j)$~~ $s\Delta F(t,x_j)u(t,x_j)\Delta s$ and

$$\Delta \mathcal{E}_p(t,x_j) = -\int_0^1 s\Delta F(t,x_j)\,u(t,x_j)\,ds$$

$$= -\tfrac{1}{2}u(t,x_j)\,\Delta F(t,x_j) \approx -T/2\, u(t,x_j)\frac{\partial^2 u}{\partial x^2}(t,x_j)\Delta x$$

$\hookrightarrow$ opposite directions of force & displacement

over all segments, $n \to \infty$

$$\mathcal{E}_p(t) = -T/2 \int_0^\ell u\frac{\partial^2 u}{\partial x^2}\,dx \underset{\text{Integration-By-Parts}}{=} T/2\int_0^\ell \left(\frac{\partial u}{\partial x}\right)^2 dx$$

The total energy is then $\mathcal{E}_p + \mathcal{E}_K$. Over a domain $\mathcal{U} \subseteq \mathbb{R}^n$, we analogously have

$$\mathcal{E}[u](t) = \frac{1}{2}\int_{\mathcal{U}} \left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2|\nabla u|^2\right] dx$$

$$\left(\text{if } c=1,\ \mathcal{E}[u](t) = \tfrac{1}{2}\int_{\mathcal{U}} |\partial_{t,x} u|^2\, dv\right)$$

$\boxed{\text{Th}^{\underline{m}}\ 4.12}$ Suppose $\mathcal{U} \subseteq \mathbb{R}^n$ is a bdd domain with piecewise C^1 boundary. If $u \in C^2([0,\infty)\times\bar{\mathcal{U}})$ solves the wave equation with $u|_{\partial\mathcal{U}} = 0$, then $\mathcal{E}[u]$ is independent of t.

$\boxed{\text{Pf}}$ $$\frac{d}{dt}\mathcal{E}[u] = \int_{\mathcal{U}} \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + c^2\nabla\left(\frac{\partial u}{\partial t}\right)\cdot\nabla u\, dx$$

Apply Green's 1^{st} ID to the second term

$$\int_{\mathcal{U}} \nabla(\partial u/\partial t)\cdot\nabla u\, dx = -\int_{\mathcal{U}} \partial u/\partial t\, \Delta u\, dx$$

(no boundary b/c $u|_{\partial\mathcal{U}} = 0$)

so $$\frac{d}{dt}\mathcal{E}[u] = \int_{\mathcal{U}}\left[\frac{\partial u}{\partial t}\underbrace{\left(\frac{\partial^2 u}{\partial t^2} - c^2\Delta u\right)}_{0}\right] dx = 0 \qquad \square$$

$\boxed{\text{Corollary}}$ Suppose $U \subset \mathbb{R}^n$ is a bdd. piecewise-C^1 domain.
A solution $u \in C^2(\mathbb{R}_{\geq 0} \times U)$ of

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f \\ u|_{\partial U} = 0 \\ u|_{t=0} = g \\ \frac{\partial u}{\partial t}\Big|_{t=0} = h \end{cases}$$

is uniquely determined by f, g, h.

$\boxed{\text{Pf}}$ Let u_1, u_2 be two solutions with the same IC & BC,

so $w = u_1 - u_2$ satisfies $\begin{cases} \frac{\partial^2 w}{\partial t^2} - c^2 \Delta w = 0 \\ w(0, x) = 0 \\ \partial_t w(0,x) = 0 \end{cases}$

This gives $\mathcal{E}[w] = 0$ for all t.
Since the integrand of $\mathcal{E}[w]$ is non-negative, it must vanish, and

$\frac{\partial w}{\partial t} = 0 = |\nabla w|$ gives w constant. The IC give

$w = 0$, so $u_1 = u_2$ $\square$